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# EFFECTS OF A MODIFICATION TO ÅNGSTRÖM'S METHOD FOR THE DETERMINATION OF THERMAL CONDUCTIVITY

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**Abstract**—A modification of the classical Ångström method for the determination of the thermal conductivity of solids is described. This modification is not necessarily desirable although at first glance it may appear to be so. A solution for the problem posed is obtained using a boundary layer type approximation to the complicated heat conduction problem. Results obtained for copper demonstrate the adequacy of the approximation.

## NOMENCLATURE

$A$, subscript denoting insulation, also cross-sectional area of specimen;
$B$, subscript denoting specimen, also amplitude of Fourier component of temperature wave;
$c$, specific heat;
$C$, amplitude of Fourier component of temperature wave;
$h$, surface heat-transfer coefficient;
$k$, thermal conductivity;
$L$, distance between measuring stations;
$P$, perimeter of specimen;
$t$, temperature;
$T$, period of temperature oscillation;
$x$, length measure in specimen;
$y$, length measure in insulation normal to surface of specimen;
$\alpha$, thermal diffusivity;
$\beta - \gamma$, phase shift;
$\tau$, time;
$\omega$, frequency of temperature oscillation;
$\theta$, temperature difference.

## INTRODUCTION

IN 1860, Ångström [1] published the results of a study of the thermal conductivity of solids as determined by an unsteady technique developed specifically for that purpose. Subsequently, many investigations have employed Ångström's method in the determination of thermal properties. Three of the most recent papers [2, 3, 4] on the subject are listed at the end of this paper.

Briefly, Ångström's method consists of the following: one end of a bar of the material whose thermal conductivity is desired, is subjected to alternate heating and cooling such that a temperature oscillation is set up in the bar. After the initial transients have died out, the temperature oscillation approaches a steady state such that the waveform of temperature at any given point along the bar reproduces *ad infinitum* with a fundamental period equal to the period of the heating and cooling cycle on the end of the bar. Ångström was able to show that from measurements of these waveforms at two different points on the bar one could determine the thermal diffusivity of the solid material. The beauty of his method lies in the fact that one needs only the amplitude and phase shift of a single Fourier component of the waveform at the two locations. The external conditions (heat-transfer coefficient and ambient temperature) are not required in the calculation as long as they are the same at both locations. In fact, the temperature of the bar is needed only to fix the level at which the diffusivity is measured, so that if thermocouples are used whose response is linear with temperature, they need not be calibrated nor must the coefficients of the Fourier components be reduced to the units of temperature. The essential simplicity of unsteady state measurements is elegantly demonstrated by this method since the thermal diffusivity having dimensions of area per unit time is simply related to a length squared, the fundamental period of the disturbance, and a dimensionless parameter calculated from the measured waveforms.

The modification discussed herein is that caused by encasing the bar of material in thermal insulation. This step at first glance seems desirable, since then a more reproducible boundary condition is obtained. In the analysis, however, we show that the thermal properties of the encasing materials now influence the results. This conclusion is a result of the interesting fact that with the bar losing heat to the atmosphere, the heat loss from the sides is nearly in phase with the local temperature excursions of the bar. With the bar encased, the heat loss lags the temperature excursion of the bar by up to 90°, forcing one to take into account the heat capacity of the insulation. The results quoted for copper both with the bar encased and open to the atmosphere can be brought into agreement with the aid of a simple theory based on a boundary-layer type approximation for the temperature field in the insulation.

## ANALYSIS

Consider the sketch shown in Fig. 1. The specimen whose thermal diffusivity is to be measured is denoted by $B$. The encasing insulation is denoted by $A$. For the purpose of the analysis, we may regard the composite structure as a sandwich, with two large blocks of insulation $A$ placed on both sides of a thin slab $B$.

If the exposed surface of a semi-infinite solid of thermal diffusivity $\alpha$ is subjected to a sinusoidal temperature variation of period $T$, the amplitude of the temperature excursion at a depth $x$ compared to the temperature excursion at the surface is given by

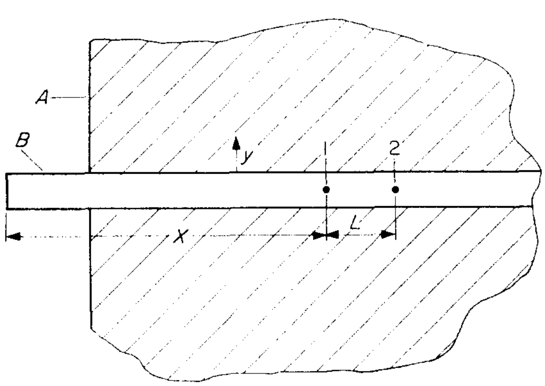


FIG. 1.

$$\theta_x/\theta_0 = \exp\left[-x\sqrt{(\pi/\alpha T)}\right]$$

If the insulation and the specimen are considered separately, the ratio of the amplitudes of the temperature excursions at a depth $x$ for the same surface temperature variation is given by $\theta_{xA}/\theta_{xB}=\exp\left[-x(\sqrt{\pi/T})(\sqrt{\alpha_B}-\sqrt{\alpha_A})/\sqrt{\alpha_A\alpha_B}\right]$. In the case considered here, the specimen was copper and the insulation asbestos. Using the values from the experiment ($x = 4$ in, $T = 4$ min, $\alpha_A = 0{\cdot}1$ ft$^2$/h, and $\alpha_B = 4{\cdot}5$ ft$^2$/h), we calculate for the ratio $\theta_{xA}/\theta_{xB}$, the value $\exp(-6{\cdot}67)$, a number vanishingly small. This simple calculation indicates that when the materials considered are placed together in the sandwich form shown in Fig. 1, the heat flow process in the insulation a short distance from the surface will be related only to the local temperature excursions of the bar and not to the surface temperature excursions of the insulation.

The foregoing calculation is not sufficient justification for the contemplated approximation: a moment's reflection reveals that the approximation really requires that the depth of penetration of the temperature excursions in the insulation normal to the encased surface of the specimen be small compared to the wavelength of the axial temperature profile in the specimen. In the insulation, the temperature excursions will have decayed to 1 per cent of their original value in a depth $Y = 4{\cdot}6\sqrt{(\alpha_A T/\pi)}$, while the wavelength of the temperature profile in the specimen is given by $X = 2\sqrt{(\pi\alpha_B T)}$. From the data given above, we calculate $Y/X = 0{\cdot}11$.

We are thus led to consider the following set of differential equations:

$$\frac{1}{\alpha_B}\frac{\partial t_B}{\partial\tau} = \frac{\partial^2 t_B}{\partial x^2} + \frac{P}{A}\frac{k_A}{k_B}\left(\frac{\partial t_A}{\partial y}\right)_{y=0} \qquad (1)$$

and

$$\frac{1}{\alpha_A}\frac{\partial t_A}{\partial\tau} = \frac{\partial^2 t_A}{\partial y^2} \qquad (2)$$

where $t_B = t_B(x, \tau)$ with $t_B(0, \tau)$ a prescribed periodic function,

$$t_A = t_A(y, \tau) \text{ with } t_A(0, \tau) = t_B(x, \tau).$$

The ratio $P/A$ is the familiar wetted perimeter to cross-sectional area ratio.* The last term in equation (1) represents the heat flow into the insulation from the sides of the specimen. It will be noted that we have made two one-dimensional approximations, in different directions, in the specimen and the insulation.

Since the temperature at any point in either the specimen or the insulation will be periodic in time, we may write the solutions to equations (1) and (2) as a Fourier series. Thus for equation (2), the well-known [5] solution is

$$t_A(y, \tau) = \sum_{n=-\infty}^{\infty} C_n \exp\left\{-y\sqrt{\left(\frac{n\omega}{2\alpha_A}\right)} - i\left[n\omega\tau - y\sqrt{\left(\frac{n\omega}{2\alpha_A}\right)}\right]\right\} \quad (3)$$

where the $C_n$ are functions of $x$ to be determined. From the boundary conditions on equations (1) and (2), we must have

$$t_B(x, \tau) = t_A(0, \tau) = \sum_{-\infty}^{\infty} C_n \exp[-in\omega\tau] \quad (4)$$

so that equation (1) becomes

$$\frac{d^2C_n}{dx^2} = \left\{\frac{P}{A}\frac{k_A}{k_B}\sqrt{\left(\frac{n\omega}{2\alpha_A}\right)} - \frac{in\omega}{\alpha_B}\left[1 + \frac{P\alpha_B k_A}{2A k_B}\sqrt{\left(\frac{2}{n\omega\,\alpha_A}\right)}\right]\right\} C_n = 0 \quad (5)$$

The solution of equation (5), though complicated algebraically, is easily written down. The nature of the problem is better elucidated, however, by a different tack. Let us assume that the $x$ values of interest are sufficiently large that the higher harmonics represented in equation (4) and (5) can be neglected. Then, without further loss of generality, the solution for the temperature field in the insulation can be written

$$t_A(y, \tau) = a(x)\exp[-y\sqrt{(\omega/2\alpha_A)}] \cos\left[\omega\tau + b(x) - y\sqrt{\left(\frac{\omega}{2\alpha_A}\right)}\right] \quad (6)$$

where $a(x)$ and $b(x)$ are functions to be determined from the solution for $t_B(x, \tau)$. From equation (6), we find

$$\frac{\partial t_A}{\partial y} = -\left[t_A + \frac{1}{\omega}\frac{\partial t_A}{\partial \tau}\right]\sqrt{\left(\frac{\omega}{2\alpha_A}\right)}$$

so that

$$\left.\frac{\partial t_A}{\partial y}\right|_{y=0} = -\left[t_B + \frac{1}{\omega}\frac{\partial t_B}{\partial \tau}\right]\sqrt{\left(\frac{\omega}{2\alpha_A}\right)}. \quad (7)$$

Inserting equation (7) into equation (1) then gives

$$k_B\frac{\partial^2 t_B}{\partial x^2} = \left\{\rho_B c_B + \rho_A c_A \frac{P}{A}\sqrt{\left(\frac{\alpha_A}{2\omega}\right)}\right\}\frac{\partial t_B}{\partial \tau} + \left\{\rho_A c_A \frac{P}{A}\sqrt{\left(\frac{\omega\alpha_A}{2}\right)}\right\} t_B \quad (8)$$

in which form the meaning of the various terms is readily apparent.

To help interpret equation (8), we may refer to Ångströms original equation, which in the present notation can be written,

$$k_B\frac{\partial^2 t_B}{\partial x^2} = \rho_B c_B\frac{\partial t_B}{\partial \tau} + \frac{hP}{A}t_B \quad (9)$$

where $h$ is a Newtonian heat-transfer coefficient.

Comparison of equations (8) and (9) now reveals that the effect of the encasing insulation is to augment the heat capacity of the material $A$ and to alter the form of the coefficient of $t_B$. Equation (8) has the same formal solution as equation (9), so that the expression for the thermal diffusivity can be taken from (5). It is,

$$\alpha_A = \frac{\left[1 + \frac{\rho_A c_A}{\rho_B c_B}\frac{P}{A}\sqrt{\left(\frac{\alpha_A}{2\omega}\right)}\right]\omega L^2}{2(\beta - \gamma)\ln B/C} \quad (10)$$

where $L$ is the distance between measuring stations along the specimen, $\beta - \gamma$ and $B/C$ are the phase shift and amplitude ratio of the temperature oscillations at the measuring stations. Equation (10) differs from the solution to equation (9), only in the presence of the second term in brackets in the numerator.

* It is not necessary for the validity of equation (1), that the specimen-insulated structure to take the geometry of a sandwich, if the radius of curvature of the surface of the specimen is small compared to the depth of penetration in the insulation of the temperature oscillation.

## EXPERIMENT

To check the above analysis, experiments were performed on an existing apparatus at Princeton University. This apparatus, regularly used as an undergraduate laboratory experiment, consists of a $\frac{1}{2}$ in dia. copper bar 3 ft long suitably mounted and instrumented with thermocouples at 4 in intervals along its length. One end of the copper bar is subjected to alternate heating and cooling from a laboratory burner and a water spray controlled by a motor and cam arrangement. The period of heating and cooling is about 4 min and the end of the bar undergoes a temperature excursion from 50°F to 350°F approximately.

Experiments were performed both with and without insulation. The insulation used consisted of a layer, about $\frac{1}{2}$ in thick, of tightly wrapped asbestos cord covered with about 1 in of corrugated asbestos pipe insulation. The estimated thermal properties of the insulation were $a_A = 0{\cdot}1$ ft²/h and $k_A = 0{\cdot}16$ Btu/h ft degF.

Temperature profiles at two locations on the specimen were recorded with a Brown strip chart recorder. The amplitude and phase of the fundamental component of the temperature profile were determined by application of the 12-ordinate method of Fourier analysis (see [6] for example).

The results can be summarized as follows:

1. No insulation, $a_B = 4{\cdot}53$ ft²/h.
2. With insulation, uncorrected, $a_B = 3{\cdot}56$ ft²/h.
3. With insulation, corrected, $a_B = 4{\cdot}10$ ft²/h.
4. Handbook value [7], $a_B = 4{\cdot}35$ ft²/h.

## CONCLUSIONS

Perhaps the most meaningful conclusion to be drawn from the above is a plea for caution in the execution of such experiments. From equation (10), we learn that the combination of parameters

$$\frac{\rho_A c_A}{\rho_B c_B} \frac{P}{A} \sqrt{\frac{a_A}{2\omega}}$$

should be small compared to unity if Ångström's method is to be employed with impunity.

## ACKNOWLEDGEMENT

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**Résumé**—On décrit une modification de la méthode classique d'Ångström pour la détermination de la conductivité thermique de solides. Cette modification n'est pas nécessairement désirable quoiqu'à première vue il puisse paraître qu'elle le soit. Une solution du problème posé est obtenue en utilisant une approximation du type de la couche limite au problème compliqué de conduction de la chaleur. Les résultats obtenus pour le cuivre démontre la justesse de l'approximation.

**Zusammenfassung**—Eine Modifikation der klassischen Ångström-Methode zur Bestimmung der Wärmeleitfähigkeit fester Körper wird beschrieben. Diese Modifikation ist nicht immer empfeh enswert, obwohl ein erster Anschein für sie spricht. Eine Lösung des komplizierten Wärmeleitungsproblems erhält man mit einer Grenzschichtnäherung. Für Kupfer erhaltene Ergebnisse zeigen die Gültigkeit der Näherung.

**Аннотация**—Описывается модификация классического метода Ангстрема для определения коэффициента теплопроводности твердых тел. Эта модификация не всегда желательна, хотя с первого взгляда может показаться обратное. Решение поставленной задачи получаем путём применения приближения пограничного слоя к сложной задаче теплопроводности. Полученные результаты для меди показывают адекватность приближения.